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PH/PH/1 Bulk Arrival and Bulk Service Queue with Randomly Varying Environment

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Abstract: *This paper studies two stochastic bulk arrival and bulk service PH/PH/1 queue Models (A) and (B) with randomly varying k* distinct environments. The arrival and service distributions are* (α_i, T_i) *and* (β_i, S_i) *in the environment i for* $1 \le i \le k^*$ *respectively. Whenever the environment changes from i to j the arrival PH and service PH distributions change from the i version to the j version with the exception of the first remaining arrival time and first remaining service time have stationary PH distributions of the j version which is known as equilibrium PH distribution for* $1 \le i, j \le k^*$ *and on completion of the same the arrival and service distributions become initial versions* (α_j, T_j) *and* (β_j, S_j)*. The queue system has infinite storing capacity and the state space is identified as five dimensional one to apply Neuts' matrix methods. The arrivals and the services occur whenever absorptions occur in the corresponding PH distributions. The sizes of the arrivals and the services are finite valued discrete random variables with distinct distributions with respect to environments and with respect to PH phases from which the absorptions occur. Matrix partitioning method is used to study the models. In Model (A) the maximum of the arrival sizes is greater than the maximum of the service sizes and the infinitesimal generator is partitioned mostly as blocks of the sum of the products of PH arrival and PH service phases in the various environments times the maximum of the arrival sizes for analysis. In Model (B) the maximum of the arrival sizes is less than the maximum of the service sizes. The generator is partitioned mostly using blocks of the same sum-product of phases times the maximum of the service sizes. Block circulant matrix structure is noticed in the basic system generators. The stationary queue length probabilities, its expected values, its variances and probabilities of empty levels are derived for the two models using matrix methods. Numerical examples are presented for illustration.*

Keywords: *Bulk Arrivals, Bulk Service, Block Circulant Matrix, Neuts Matrix Methods, Phase Type Distribution, Stationary PH Distribution.*

I. INTRODUCTION

In this paper two bulk arrival and bulk service PH/PH/1 queues with random environment have been studied using matrix geometric methods. Numerical studies on matrix methods are presented by Bini, Latouche and Meini [1]. Multi server model has been of interest in Chakravarthy and Neuts [2]. Birth and death model has been analyzed by Gaver, Jacobs and Latouche [3]. Analytic methods are focused in Latouche and Ramaswami [4] and for matrix geometric methods one may refer Neuts [5]. For M/M/1 bulk queues with random environment models one may refer Rama Ganesan, Ramshankar and Ramanarayanan [6] and M/M/C bulk queues with random environment models are of interest in Sandhya, Sundar, Rama, Ramshankar and Ramanarayanan [7]. PH/PH/1 bulk queues without variation of environments have been treated by Ramshankar, Rama Ganesan and Ramanarayanan [8]. The models considered here are general compared to existing models. Random number of arrivals and random number of services are considered at a time. Fixed numbers of customers are cleared by a service in the models of Neuts and Nadarajan [9]. In real life situations when a machine manufactures a fixed number of products in every production schedule, the defective items are rejected in all production lots', making the production lot is only of random size and not a fixed one always. Situations of random bulk services are seen often in software based industries where finished software projects waiting for marketing are sold in bulk sizes when there is economic boom and the business may be insignificant when there is economic recession. In industrial productions, bulk types are very common. Manufactured products arrive in various bulk sizes for sale in markets and the products are sold in various bulk sizes depending on market requirements. Noam Paz and Uri Yechali [10] have studied M/M/1 queue with disaster. Usually bulk arrival models have M/G/1 upper-Heisenberg block matrix structure. The decomposition of a Toeplitz sub matrix of the infinitesimal generator is required to find the stationary probability vector as done in William J. Stewart [11] and matrix geometric structures have not been noted. In such models the recurrence relation method to find the stationary probabilities is stopped at a certain level in most general cases using a terminating analysis very well explained by Qi-Ming He [12] and this stopping limitation of terminating method converts an infinite arrival system to a finite arrival one. In special cases generating function has been identified by Rama and Ramanarayanan [13]. However the division modulo partitioning of the infinitesimal generator along with environment and PH phases used in the paper is presenting matrix geometric solution for finite sized arrivals and services models. The M/PH/1 and PH/M/C queues with random environments have been studied by Usha [14] and [15] without bulk arrivals and bulk services. It has been noticed by Usha [14, 15] that when the environment changes the remaining arrival and service times are to be completed in the new environment. The residual arrival time and the residual service time distributions in the new environment are to be considered in the new environment at an arbitrary epoch since the spent arrival time and the spent service time have been in the previous environment with distinct sizes of PH phase. Further new arrival time and new service time from the start using initial PH distributions of the new environment cannot be considered since the arrival and the service have been partly completed in the previous environment indicating the stationary versions of the arrival and service distributions in the new environments are to be used for the completions of the residual arrival and service times in the new environment and on completion of the same the next arrival and service onwards they have initial versions of the PH distributions of the new environment. The stationary version of the distribution for residual time has been well explained in Qi-Ming He [12] where it is named as equilibrium PH distribution. Randomly varying environment PH/PH/1 queue models with bulk arrival and bulk service have not been treated so far at any depth. In this paper the partitioning of the matrix is carried out in a way that the stationary probability vector exhibits a matrix geometric structure for PH/PH/1 bulk queues with random environment where the arrivals and service sizes are finite. Two models (A) and (B) on PH/PH/1 bulk queue systems with infinite storage space for customers are studied using the block partitioning method. Model (A) presents the case when M, the maximum of the arrival sizes is bigger than N, the maximum of the service sizes. In Model (B), its dual case N is bigger than M, is treated. In general in Queue models, the state space of the system has the first co-ordinate indicating the number of customers in the system but here the customers in the system are grouped and considered as members of blocks of sizes of the maximum for finding the rate matrix. Using the maximum of the bulk arrival size or the maximum of the bulk service size and grouping the customers as members of blocks in addition to coordinates of the arrival and service phases for the partitioning the infinitesimal generator is a new approach in this area. The matrices appearing as the basic system generators in these two models due to block partitioned structure are seen as block circulant matrices. The paper is organized in the following manner. In sections II and III the stationary probability of the number of customers waiting for service, the expectation and the variance and the probability of empty queue are derived for these Models (A) and (B). In section IV numerical cases are presented to illustrate them.

II.MODEL (A): MAXIMUM ARRIVAL SIZE M > MAXIMUM SERVICE SIZE N

2.1Assumptions

(i)There are k* environments. The environment changes as per changes in a continuous time Markov chain with infinitesimal generator Q_1 of order k* with stationary probability vector π'.

(ii)In the environment i for 1 ≤ i ≤ k*, the time between consecutive epochs of bulk arrivals of customers has phase type distribution (α_i, T_i) where $T_i$is a matrix of order k_i with absorbing rate $T'_i = -T_i e$ to the absorbing state k_i+1 from where the arrival process moves instantaneously to a starting state as per the starting vector α_i = $(\alpha_{i,1}, \alpha_{i,2}., ..., \alpha_{i,k_i})$ and $\sum_{j=1}^{k_i} \alpha_{i,j}$ = 1. Let φ_i be the invariant probability vector of the generator matrix $(T_i + T'_i\ \alpha_i)$.

(iii) In the environment i for 1 ≤ i ≤ k*,when the absorption occurs in the PH arrival process due to transition from a state j to state k_i +1, ${}^{i}_{j}\chi$ number of customers arrive with probabilities P (${}^{i}_{j}\chi$= n) = ${}^{i}_{j}p_n$for 1 ≤ n ≤ ${}^{i}_{j}M$ and $\sum_{n=1}^{{}^{i}_{j}M} {}^{i}_{j}p_n$=1where ${}^{i}_{j}M$ is the maximum arrival size for PH phase j where 1≤ j ≤ k_i .

(iv) In the environment i for 1 ≤ i ≤ k*, the time between consecutive epochs of bulk services of customers has phase type distribution (β_i, S_i) where S_i is a matrix of order k'_i with absorbing rate $S'_i = -S_i e$ to the absorbing state k'_i+1 from where the service process moves instantaneously to a starting state as per the staring vector β_i = $(\beta_{i,1}, \beta_{i,2}., ..., \beta_{i,k'_i})$ and $\sum_{j=1}^{k'_i} \beta_{i,j}$ = 1. Let ϕ_i be the invariant probability vector of the generator matrix $(S_i + S'_i\ \beta_i)$.

(v) In the environment i for1 ≤ i ≤ k*, customers of bulk size ${}^{i}_{j}\psi$ are served at epoch when the absorption occurs due to a transition from state j to state k'_i+1, with probabilities P(${}^{i}_{j}\psi$ = n) = ${}^{i}_{j}q_n$ for 1≤ n ≤ ${}^{i}_{j}N$ and $\sum_{j=1}^{{}^{i}_{j}N} {}^{i}_{j}q_n$ = 1, when more than ${}^{i}_{j}N$ customers are waiting for service where ${}^{i}_{j}N$ is the maximum service size for PH phase j where $1 \le j \le k'_i$. When n customers n < ${}^i_jN$ are waiting for service, then n' customers are served with probability ${}^i_jq_{n'}$ for 1≤ n' ≤ n-1 and n customers are served with probability $\sum_{j=n}^{{}^i_jN} {}^i_jq_n$ for PH phase j where $1 \le j \le k'_i$.

(vi) When the environment changes from i to j for 1 ≤ i, j ≤ k*, the arrival and service distributions in the new environment j are the stationary (equilibrium)versions of arrival time and service time distributions in the new environment, namely, (φ_j , T_j) and (ϕ_j , S_j) respectively for the completions of the residual arrival and service times and on completion of the same the next arrival and service onwards they have initial versions of the PH distributions of the new environment namely (α_j , T_j) and (β_j , S_j) respectively.

(vii) The maximum arrival size M= $max_{1 \le i \le k*} \max_{1 \le j \le k_i} {}^i_jM$ is greater than the maximum service size N= $max_{1 \le i \le k*} \max_{1 \le j \le k'_i} {}^i_jN$.

2.2.Analysis

The state of the system of the continuous time Markov chain X (t) under consideration is presented as follows.
X(t) = {(0, i, j) : for 1 ≤ i ≤ k* 1 ≤ j ≤ k_i)} U {(0, k, i, j, j') ; for 1 ≤ k ≤ M-1; 1 ≤ i ≤ k*; 1 ≤ j ≤ k_i; 1 ≤ j ≤ k'_i}
U {(n, k, i, j, j'): for 0 ≤ k ≤ M-1; 1 ≤ i ≤ k*; 1 ≤ j ≤ k_i; 1 ≤ j ≤ k'_i and n ≥ 1}. (1)

The chain is in the state (0, i, j) when the number of customers in the queue is 0, the environment state is i for 1 ≤ i ≤ k*and the arrival phase is j for 1 ≤ j ≤ k_i. The chain is in the state (0, k, i, j, j') when the number of customers is k for 1 ≤ k ≤ M-1, the environment state is i for 1 ≤ i ≤ k*, the arrival phase is j for 1 ≤ j ≤ k_i and the service phase is j' for 1 ≤ j' ≤ k'_i. The chain is in the state (n, k, i, j, j') when the number of customers in the queue is n M + k, for 0 ≤ k ≤ M-1 and 1 ≤ n < ∞, the environment state is i for 1 ≤ i ≤ k*, the arrival phase is j for 1 ≤ j ≤ k_i and the service phase is j' for 1 ≤ j' ≤ k'_i. When the number of customers waiting in the system is r, then r is identified with (n, k) where r on division by M gives n as the quotient and k as the remainder. Let the survivor probabilities of arrivals ${}^i_j\chi$ and of services ${}^i_j\psi$ be respectively P(${}^i_j\chi$>m)= ${}^i_jP_m$ =1-$\sum_{n=1}^{m} {}^i_jp_n$,for
1 ≤ m ≤ ${}^i_jM$ -1 and 1≤ j ≤ k_i (2)

P(${}^i_j\psi$>m)= ${}^i_jQ_m$=1-$\sum_{n=1}^{m} {}^i_jq_n$, for 1 ≤ m ≤ ${}^i_jN$ -1and 1≤ j ≤ k'_i (3)

with ${}^i_jP_0 = 1$, for all j, 1≤ j ≤ k_i and ${}^i_jQ_0 = 1$ for all j , 1≤ j ≤ k'_i for the environment state i for 1 ≤ i ≤ k* .
The chain X (t) describing model has the infinitesimal generator Q_A of infinite order which can be presented in block partitioned form given below.

$$Q_A=\begin{bmatrix} B_1 & B_0 & 0 & 0 & . & . & . & \cdots \\ B_2 & A_1 & A_0 & 0 & . & . & . & \cdots \\ 0 & A_2 & A_1 & A_0 & 0 & . & . & \cdots \\ 0 & 0 & A_2 & A_1 & A_0 & 0 & . & \cdots \\ 0 & 0 & 0 & A_2 & A_1 & A_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \quad (4)$$

In (4) the states of the matrices are listed lexicographically as $\underline{0}, \underline{1}, \underline{2}, \underline{3}$, ... For partition purpose the zero states in the first two sets given in (1) are combined. The vector $\underline{0}$ is of type 1 x $[\sum_{i=1}^{k*} k_i + (M-1)\sum_{i=1}^{k*} k_i\, k'_i]$ and is $\underline{0}$=((0,1,1),(0,1,2),(0,1,3)…(0,1, k_1),(0,2,1),(0,2,2),(0,2,3)…(0,2, k_2),……(0,k*,1),(0,k*,2),(0,k*,3)…(0,k*,k_{k*}), (0,1,1,1,1),(0,1,1,1,2)….(0,1,1,1, k'_1),(0,1,1,2,1),(0,1,1,2,2)….(0,1,1,2,k'_1),(0,1,1,3,1)....(0,1,1,3,k'_1)…..(0,1,1, k_1,1)…(0,1,1,k_1, k'_1),(0,1,2,1,1),(0,1,2,1,2)….(0,1,2,1, k'_2),(0,1,2,2,1),(0,1,2,2,2)….(0,1,2,2,k'_2),(0,1,2,3,1)....(0,1,2,3,k'_2)….(0,1,2,k_2,1)…(0,1,2,k_2, k'_2),(0,1,3,1,1)…(0,1,3,k_3, k'_3)…(0,1,k*,1,1),…,(0,1,k*,k_{k*}, k'_{k*}),(0,2,1,1,1),(0,2,1,1,2),…,(0,2,k*,k_{k*},k'_{k*}),(0,3,1,1,1)…(0,3,k*,$k_{k*}k_{k*}$),(0,4,1,1,1)…(0,4,k*,$k_{k*}k'_{k*}$)…(0,M-1,1,1,1)…(0,M-1,k*,k_{k*}, k'_{k*})) and the vector $\underline{n}$ is of type 1x$[M\sum_{i=1}^{k*} k_i\, k'_i]$ and is given in a similar manner as follows $\underline{n}$=(n,0,1,1,1),(n,0,1,1,2)….(n,0,1,1, k'_1),(n,0,1,2,1),…(n,0,1,2,k'_1),…(n,0,k*,1,1),…(n,0,k*,k_{k*}, k'_{k*}),(n,1,1,1,1)….(n,1,k*,k_{k*},k'_{k*}),(n,2,1,1,1)....(n,2,k*,k_{k*},k'_{k*})……..(n,M-1,1,1,1),(n,M-1,1,1,2)……(n,M-1,k*,k_{k*}, k'_{k*})).

The matrices B_1 *and* A_1 have negative diagonal elements, they are of orders $[\sum_{i=1}^{k*} k_i + (M-1)\sum_{i=1}^{k*} k_i\, k'_i]$ and $[M\sum_{i=1}^{k*} k_i\, k'_i]$ respectively and their off diagonal are non-negative. The matrices A_0 *and* A_2 have nonnegative elements and are of order $[M\sum_{i=1}^{k*} k_i\, k'_i]$. The matrices B_0 *and* B_2 have non-negative elements and are of types $[\sum_{i=1}^{k*} k_i + (M-1)\sum_{i=1}^{k*} k_i\, k'_i]$ x $[M\sum_{i=1}^{k*} k_i\, k'_i]$ and $[M\sum_{i=1}^{k*} k_i\, k'_i]$ x $[\sum_{i=1}^{k*} k_i + (M-1)\sum_{i=1}^{k*} k_i\, k'_i]$. Component matrices of A_i *and* B_i for i=0,1,2 are defined below. Let ⊕ *and* ⊗ denote the Kronecker sum and Kronecker product.

Let $Q'_i = T_i \oplus S_i$ + diag $((Q_1)_{i,i})$ = $(T_i \otimes I_{k'_i}) + (I_{k_i} \otimes S_i)$ + diag $((Q_1)_{i,i})$ for 1 ≤ i ≤ k* (5)

where I indicates the identity matrices of orders given in the suffixes, Q'_i is of order $k_i k'_i$ and the last term is a diagonal matrix of order $k_i k'_i$. Considering the change of environment switches on stationary distributions in PH arrival time and PH service time in the new environment, the following matrix Ω of order $\sum_{i=1}^{k*} k_i\, k'_i$ is defined which is concerned with change of environment during arrival time and service time.

$$\Omega=\begin{bmatrix} Q'_1 & \Omega_{1,2} & \Omega_{1,3} & \cdots & \Omega_{1,k*} \\ \Omega_{2,1} & Q'_2 & \Omega_{2,3} & \cdots & \Omega_{2,k*} \\ \Omega_{3,1} & \Omega_{3,2} & Q'_3 & \cdots & \Omega_{3,k*} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Omega_{k*,1} & \Omega_{k*,2} & \Omega_{k*,3} & \cdots & Q'_{k*} \end{bmatrix} \quad (6)$$

where $\Omega_{i,j}$ is a rectangular matrix of type $k_i k'_i$ x $k_j k'_j$ whose all rows are equal to $(Q_1)_{i,j}$ $(\varphi_j \otimes \phi_j)$ for i ≠ j , 1 ≤ i, j ≤ k*. The arrival rate of n customers for 1≤ n ≤ ${}^i_jM$ corresponding to absorption to state k_i+1 from the arrival PH phase j for 1≤ j ≤ k_i, in the environment i for 1≤ i ≤ k* is given by the component j of the column vector $T'_{i,n}$ of type k_i x1 where $T'_{i,n}$= ($(T'_i)_1({}^i_1p_n)$, $(T'_i)_2({}^i_2p_n)$, $(T'_i)_3({}^i_3p_n)$, ….$(T'_i)_{k_i}({}^i_{k_i}p_n)$)'; (7)
the service rate of n customers for 1≤ n ≤ ${}^i_jN$ corresponding to absorption to state k'_i+1 from the service PH phase j for 1≤ j ≤ k'_i, in the environment i for 1≤ i ≤ k* is given by the component j of the column vector $S'_{i,n}$ of type k'_i x1where $S'_{i,n}$= ($(S'_i)_1({}^i_1q_n)$, $(S'_i)_2({}^i_2q_n)$, $(S'_i)_3({}^i_3q_n)$, ….$(S'_i)_{k'_i}({}^i_{k'_i}q_n)$)' . (8)

$$\text{Let } \Lambda_n = \begin{bmatrix} T'_{1,n}\,\alpha_1 \otimes I_{k'_1} & 0 & 0 & \cdots & 0 \\ 0 & T'_{2,n}\,\alpha_2 \otimes I_{k'_2} & 0 & \cdots & 0 \\ 0 & 0 & T'_{3,n}\,\alpha_3 \otimes I_{k'_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & T'_{k*,n}\,\alpha_{k*} \otimes I_{k'_{k*}} \end{bmatrix} \text{ for } 1 \le n \le M \quad (9)$$

In (9) Λ_n is a square matrix of order $\sum_{i=1}^{k*} k_i\, k'_i$; $T'_{j,n}\ \alpha_j \otimes I_{k'_j}$ is a square matrix of order $k_j k'_j$ for 1 ≤ j ≤ k* and 0 appearing as (i, j) component of (9) is a block zero rectangular matrix of type $k_i k'_i$ x $k_j k'_j$.

$$\text{Let } U_n = \begin{bmatrix} I_{k_1} \otimes S'_{1,n}\,\beta_1 & 0 & 0 & \cdots & 0 \\ 0 & I_{k_2} \otimes S'_{2,n}\,\beta_2 & 0 & \cdots & 0 \\ 0 & 0 & I_{k_3} \otimes S'_{3,n}\,\beta_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_{k_{k*}} \otimes S'_{k*,n}\,\beta_{k*} \end{bmatrix} \text{ for } 1 \le n \le N \quad (10)$$

In (10) U_n is a square matrix of order $\sum_{i=1}^{k*} k_i\, k'_i$; $I_{k_j} \otimes S'_{j,n}\ \beta_j$ is a square matrix of order $k_j k'_j$ for 1 ≤ j ≤ k* and 0 appearing as (i, j) component of (10) is a block zero rectangular matrix of type $k_i k'_i$ x $k_j k'_j$ .The matrix A_i for i = 0,1,2 are as follows.

$$A_0 = \begin{bmatrix} \Lambda_M & 0 & \cdots & 0 & 0 & 0 \\ \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0 & 0 \\ \Lambda_{M-2} & \Lambda_{M-1} & \cdots & 0 & 0 & 0 \\ \Lambda_{M-3} & \Lambda_{M-2} & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \Lambda_3 & \Lambda_4 & \cdots & \Lambda_M & 0 & 0 \\ \Lambda_2 & \Lambda_3 & \cdots & \Lambda_{M-1} & \Lambda_M & 0 \\ \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-2} & \Lambda_{M-1} & \Lambda_M \end{bmatrix} (11) \qquad A_2 = \begin{bmatrix} 0 & \cdots & 0 & U_N & U_{N-1} & \cdots & U_2 & U_1 \\ 0 & \cdots & 0 & 0 & U_N & \cdots & U_3 & U_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & U_N & U_{N-1} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & U_N \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} (12)$$

$$A_1 = \begin{bmatrix} \Omega & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \Lambda_{M-N} & \cdots & \Lambda_{M-2} & \Lambda_{M-1} \\ U_1 & \Omega & \Lambda_1 & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \cdots & \Lambda_{M-3} & \Lambda_{M-2} \\ U_2 & U_1 & \Omega & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \cdots & \Lambda_{M-4} & \Lambda_{M-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ U_N & U_{N-1} & U_{N-2} & \cdots & \Omega & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} \\ 0 & U_N & U_{N-1} & \cdots & U_1 & \Omega & \Lambda_1 & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} \\ 0 & 0 & U_N & \cdots & U_2 & U_1 & \Omega & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & U_N & U_{N-1} & U_{N-2} & \cdots & \Omega & \Lambda_1 \\ 0 & 0 & 0 & \cdots & 0 & U_N & U_{N-1} & \cdots & U_1 & \Omega \end{bmatrix} \quad (13)$$

For defining the matrices B_i for i = 0,1,2 the following component matrices are required

$$\Lambda'_n = \begin{bmatrix} T'_{1,n}\,\alpha_1 \otimes \beta_1 & 0 & 0 & \cdots & 0 \\ 0 & T'_{2,n}\,\alpha_2 \otimes \beta_2 & 0 & \cdots & 0 \\ 0 & 0 & T'_{3,n}\,\alpha_3 \otimes \beta_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & T'_{k*,n}\,\alpha_{k*} \otimes \beta_{k*} \end{bmatrix} \text{ for } 1 \le n \le M \quad (14)$$

Λ'_n is a rectangular matrix of type $(\sum_{i=1}^{k*} k_i)$x $\sum_{i=1}^{k*}(k_i k'_i)$ for 1 ≤ n ≤ M ; $T'_{i,n}\ \alpha_i \otimes \beta_i$ is a rectangular matrix of order $k_i x k_i k'_i$ and 0 appearing as (i, j) component of (14) is a block zero rectangular matrix of type k_i x $k_j k'_j$

for 1 ≤ i, j ≤ k*. Let $V'_{i,n} = I_{k_i} \otimes ((S'_i)_1({}_1^iQ_n), (S'_i)_2({}_2^iQ_n), (S'_i)_3({}_3^iQ_n), \ldots (S'_i)_{k'_i}\left({}_{k'_i}^{\,i}Q_n\right))'$ for 1 ≤ n ≤ N -1 is a matrix of type $k_i k'_i$ x k_i for 1 ≤ i ≤ k*and let

$$V_n = \begin{bmatrix} V'_{1,n} & 0 & 0 & \cdots & 0 \\ 0 & V'_{2,n} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & V'_{k*,n} \end{bmatrix} \text{ for } 1 \le n \le N. \qquad (15)$$

This is a rectangular matrix of type ($\sum_{i=1}^{i=k*} k_i k'_i) \, x \left(\sum_{i=1}^{k*} k_i\right)$ and 0 appearing in the (i, j) component is a rectangular 0 matrix of type $k_i k'_i$ x k_j for 1 ≤ i, j ≤ k*.

$$\text{Let U} = \begin{bmatrix} I_{k_1} \otimes S'_1 & 0 & 0 & \cdots & 0 \\ 0 & I_{k_2} \otimes S'_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_{k_1} \otimes S'_{k*} \end{bmatrix} \qquad (16)$$

In (16), U is a rectangular matrix of type ($\sum_{i=1}^{i=k*} k_i k'_i) \, x \left(\sum_{i=1}^{k*} k_i\right)$ and 0 appearing in the (i, j) component is a rectangular 0 matrix of type $k_i k'_i$ x k_j for 1 ≤ i, j ≤ k*. $I_{k_i} \otimes S'_i$ is a rectangular matrix of type $k_i k'_i \; x \; k_i$ for 1 ≤ i ≤ k*. The matrix B_0 is same as that of A_0 when Λ_M in the first row of $A_0$is replaced byΛ'_M. The matrix B_1 is given below. The matrix B_2 is same as that of $A_2$when the first block column with 0 is considered as $\sum_{i=1}^{k*} k_i$ columns block instead of $\sum_{i=1}^{k*} k_i k'_i$columns block of A_2. To write $B_1$the block for $\underline{0}$ is to be considered which has queue length, L= 0, 1, 2…M-1. When L = 0 there is only arrival process and no service process. The change in environment from i to j switches on stationary PH (equilibrium PH) distribution in the new environment j whenever it occurs for 1 ≤ i ≠ j, ≤ k*. When an arrival occurs and queue length becomes L in the environment i both the arrival time and the service time start with starting probability vector α_i and β_i respectively for 1 ≤ i ≤ k*. In the $\underline{0}$ when L =1,2, …M-1 all the processes arrival, service and environment are active as in other blocks $\underline{n}$ for n > 0. Considering the change of environment switches on the stationary (equilibrium) distribution in PH arrival time in the new environment when the queue is empty, the following matrix Ω' of order $\sum_{i=1}^{k*} k_i$is defined which is concerned with change of environment during arrival time.

$$\Omega' = \begin{bmatrix} T'_1 & \Omega'_{1,2} & \Omega'_{1,3} & \cdots & \Omega'_{1,k*} \\ \Omega'_{2,1} & T'_2 & \Omega'_{2,3} & \cdots & \Omega'_{2,k*} \\ \Omega'_{3,1} & \Omega'_{3,2} & T'_3 & \cdots & \Omega'_{3,k*} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Omega'_{k*,1} & \Omega'_{k*,2} & \Omega'_{k*,3} & \cdots & T'_{k*} \end{bmatrix}$$

(17)

Here $T'_i = T_i + diag(Q_1)_{i,i}$ and $\Omega'_{i,j}$ is a rectangular matrix of type k_i x k_j whose all rows are equal to $(Q_1)_{i,j}\,\varphi_j$ presenting the rates of changing to phases in the new environment for i ≠ j and 1 ≤ i, j ≤ k*.

$$B_1 = \begin{bmatrix} \Omega' & \Lambda'_1 & \Lambda'_2 & \cdots & \Lambda'_{M-N-2} & \Lambda'_{M-N-1} & \Lambda'_{M-N} & \cdots & \Lambda'_{M-2} & \Lambda'_{M-1} \\ U & \Omega & \Lambda_1 & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \cdots & \Lambda_{M-3} & \Lambda_{M-2} \\ V_1 & U_1 & \Omega & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \cdots & \Lambda_{M-4} & \Lambda_{M-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ V_{N-1} & U_{N-1} & U_{N-2} & \cdots & \Omega & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} \\ 0 & U_N & U_{N-1} & \cdots & U_1 & \Omega & \Lambda_1 & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} \\ 0 & 0 & U_N & \cdots & U_2 & U_1 & \Omega & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & U_N & U_{N-1} & U_{N-2} & \cdots & \Omega & \Lambda_1 \\ 0 & 0 & 0 & \cdots & 0 & U_N & U_{N-1} & \cdots & U_1 & \Omega \end{bmatrix} \qquad (18)$$

$$Q'_A = \begin{bmatrix} \Omega+\Lambda_M & \Lambda_1 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \Lambda_{M-N}+U_N & \cdots & \Lambda_{M-2}+U_2 & \Lambda_{M-1}+U_1 \\ \Lambda_{M-1}+U_1 & \Omega+\Lambda_M & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \cdots & \Lambda_{M-3}+U_3 & \Lambda_{M-2}+U_2 \\ \Lambda_{M-2}+U_2 & \Lambda_{M-1}+U_1 & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \cdots & \Lambda_{M-4}+U_3 & \Lambda_{M-3}+U_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Lambda_{M-N+2}+U_{N-2} & . & \cdots & . & . & . & \cdots & \Lambda_{M-N}+U_N & \Lambda_{M-N+1}+U_{N-1} \\ \Lambda_{M-N+1}+U_{N-1} & . & \cdots & . & . & . & . & \Lambda_{M-N-1} & \Lambda_{M-N}+U_N \\ \Lambda_{M-N}+U_N & . & \cdots & \Omega+\Lambda_M & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} \\ \Lambda_{M-N-1} & \Lambda_{M-N}+U_N & \cdots & \Lambda_{M-1}+U_1 & \Omega+\Lambda_M & \Lambda_1 & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} \\ \Lambda_{M-N-2} & \Lambda_{M-N-1} & \cdots & \Lambda_{M-2}+U_2 & \Lambda_{M-1}+U_1 & \Omega+\Lambda_M & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Lambda_2 & \Lambda_3 & \cdots & \Lambda_{M-N}+U_N & \Lambda_{M-N+1}+U_{N-1} & \Lambda_{M-N+2}+U_{N-2} & \cdots & \Omega+\Lambda_M & \Lambda_1 \\ \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-1} & \Lambda_{M-N}+U_N & \Lambda_{M-N+1}+U_{N-1} & \cdots & \Lambda_{M-1}+U_1 & \Omega+\Lambda_M \end{bmatrix} \qquad (19)$$

The basic generator of the bulk queue which is concerned with only the arrival and service is a matrix of order $[\, M \sum_{i=1}^{k*} k_i \, k'_i \,]$ given above in (19) where $Q'_A = A_0 + A_1 + A_2$ (20)

Its probability vector w' gives, $w'Q_A' = 0$ and w'. e = 1 (21)
It is well known that a square matrix in which each row (after the first) has the elements of the previous row shifted cyclically one place right, is called a circulant matrix. It is very interesting to note that the matrix Q_A' is a block circulant matrix where each block matrix is rotated one block to the right relative to the preceding block partition. In (19), the first block-row of type $[\sum_{i=1}^{k*} k_i\, k'_i]$ x$[M\sum_{i=1}^{k*} k_i\, k'_i]$ is, $W = (\Omega + \Lambda_M, \Lambda_1, \Lambda_2, \ldots, \Lambda_{M-N-2}, \Lambda_{M-N-1}, \Lambda_{M-N} + U_N, \ldots, \Lambda_{M-2} + U_2, \Lambda_{M-1} + U_1)$ which gives as the sum of the blocks $(\Omega + \Lambda_M) + \Lambda_1 + \Lambda_2 + \ldots + \Lambda_{M-N-2} + \Lambda_{M-N-1} + \Lambda_{M-N} + U_N + \ldots + \Lambda_{M-2} + U_2 + \Lambda_{M-1} + U_1 = \Omega''$ which is the matrix given by

$$\Omega'' = \begin{bmatrix} Q''_1 & \Omega_{1,2} & \Omega_{1,3} & \cdots & \Omega_{1,k*} \\ \Omega_{2,1} & Q''_2 & \Omega_{2,3} & \cdots & \Omega_{2,k*} \\ \Omega_{3,1} & \Omega_{3,2} & Q''_3 & \cdots & \Omega_{3,k*} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Omega_{k*,1} & \Omega_{k*,2} & \Omega_{k*,3} & \cdots & Q''_{k*} \end{bmatrix} \quad (22)$$

where using (5) and (6), $Q''_i = ((T_i + T'_i\,\alpha_i)\otimes I_{k'_i}) + (I_{k_i}\otimes(S_i + S'_i\,\beta_i))$ + diag $((Q_1)_{i,i})$ for $1 \le i \le k^*$. The stationary probability vector of the basic generator given in (19) is required to get the stability condition. Consider the vector w = $(\pi'_1\varphi_1 \otimes \phi_1, \pi'_2\varphi_2 \otimes \phi_2, \ldots, \pi'_{k*}\varphi_{k*} \otimes \phi_{k*})$ where $\pi' = (\pi'_1, \pi'_2, \ldots, \pi'_{k*})$ is the stationary probability vector of the environment, φ_i *and* ϕ_i are the stationary probability vectors of the arrival and service PH processes $(T_i + T'_i\,\alpha_i)$ and $(S_i + S'_i\,\beta_i)$ respectively. It may be noted $\pi'_i(\varphi_i \otimes \phi_i)[((T_i + T'_i\,\alpha_i)\otimes I_{k'_i}) + (I_{k_i}\otimes(S_i + S'_i\,\beta_i))] = 0$. This gives $\pi'_i(\varphi_i \otimes \phi_i)Q''_i = \pi'_i(Q_1)_{i,i}\,(\varphi_i \otimes \phi_i)\,I = \pi'_i(Q_1)_{i,i}\,(\varphi_i \otimes \phi_i)$ for $1 \le i \le k^*$. Now the first column of the matrix multiplication of wΩ'' is $\pi'_1(Q_1)_{1,1}\varphi_{1,1}\phi_{1,1} + \pi'_2\,(Q_1)_{2,1}\varphi_{11}\phi_{11}[(\varphi_2 \otimes \phi_2)e] + \ldots + \pi'_{k*}\,(Q_1)_{k*,1}\varphi_{11}\phi_{11}[(\varphi_{k*} \otimes \phi_{k*})]e = 0$ since $(\varphi_i \otimes \phi_i)e = 1$ and $\pi' Q_1 = 0$. In a similar manner it can be seen that the first column block of wΩ'' is $\pi'_1(Q_1)_{1,1}\varphi_1 \otimes \phi_1 + \pi'_2\,(Q_1)_{2,1}\varphi_1 \otimes \phi_1[(\varphi_2 \otimes \phi_2)e] + \ldots + \pi'_{k*}\,(Q_1)_{k*,1}\varphi_1 \otimes \phi_1[(\varphi_{k*} \otimes \phi_{k*})]e = 0$ and i-th column block is $\pi'_1(Q_1)_{1,i}\varphi_i \otimes \phi_i[(\varphi_1 \otimes \phi_1)e] + \pi'_2\,(Q_1)_{2,i}\varphi_i \otimes \phi_i[(\varphi_2 \otimes \phi_2)e] + \ldots + \pi'_i\,(Q_1)_{i,i}\varphi_i \otimes \phi_i + \ldots + \pi'_{k*}\,(Q_1)_{k*,i}\varphi_i \otimes \phi_i[(\varphi_{k*} \otimes \phi_{k*})]e = 0$. This shows that $w(\Omega + \Lambda_M) + w\Lambda_1 + w\Lambda_2 + \ldots + w\Lambda_{M-N-2} + w\Lambda_{M-N-1} + w\Lambda_{M-N} + wU_N + \ldots + w\Lambda_{M-2} + wU_2 + w\Lambda_{M-1} + wU_1$ = w Ω''=0. So (w, w,…,w) .W= 0 = (w, w, ….w) W' where W' is the transpose W. This shows (w,w...w) is the left eigen vector of Q_A' and the corresponding probability vector is w' = $\left(\frac{w}{M}, \frac{w}{M}, \frac{w}{M}, \ldots\ldots, \frac{w}{M}\right)$ where w is given by

w = $(\pi'_1(\varphi_1 \otimes \phi_1), \pi'_2(\varphi_2 \otimes \phi_2), \ldots\ldots, \pi'_{k*}(\varphi_{k*} \otimes \phi_{k*}))$ (23)

Let $\varphi_i = (\varphi_{i,j})$ and $\phi_i = (\phi_{i,j})$ be the stationary probability components of the arrival and service processes. Neuts [5], gives the stability condition as, $w' A_0\, e < w' A_2\, e$ where w is given by (23). Taking the sum cross diagonally in the A_0 *and* A_2 matrices, it can be seen using (9) that

w' $A_0\, e = \frac{1}{M}\, w(\sum_{n=1}^{M} n\Lambda_n)e = \frac{1}{M}\left(\sum_{n=1}^{M}\sum_{i=1}^{k*} n\,\pi'_i(\varphi_i \otimes \phi_i)(T'_{i,n} \otimes e)\right) =$
$\frac{1}{M}\left(\sum_{n=1}^{M}\sum_{i=1}^{k*} n\,\pi'_i(\varphi_i T'_{i,n} \otimes \phi_i e)\right) = \frac{1}{M}\left(\sum_{n=1}^{M}\sum_{i=1}^{k*} n\,\pi'_i(\varphi_i T'_{i,n})\right) = \frac{1}{M}(\sum_{i=1}^{k*}\pi'_i\sum_{n=1}^{M} n\sum_{j=1}^{k_i}\varphi_{i,j}\,(T'_{i,n})_j)$
$= \frac{1}{M}(\sum_{i=1}^{k*}\pi'_i\sum_{n=1}^{M} n\sum_{j=1}^{k_i}\varphi_{i,j}\,(T'_i)_j\,({}^i_j p_n)) = \frac{1}{M}(\sum_{i=1}^{k*}\pi'_i\sum_{j=1}^{k_i}\varphi_{i,j}\,(T'_i)_j\, E({}^i_j\chi) < w' A_2\, e$
$= \frac{1}{M}\, w(\sum_{n=1}^{N} nU_n)e = \frac{1}{M}\left(\sum_{n=1}^{N}\sum_{i=1}^{k*} n\,\pi'_i(\varphi_i \otimes \phi_i)(e \otimes S'_{i,n})\right) = \frac{1}{M}\left(\sum_{n=1}^{N}\sum_{i=1}^{k*} n\,\pi'_i(\varphi_i e \otimes \phi_i S'_{i,n})\right)$
$= \frac{1}{M}\left(\sum_{n=1}^{N}\sum_{i=1}^{k*} n\,\pi'_i(\phi_i S'_{i,n})\right) = \frac{1}{M}(\sum_{i=1}^{k*}\pi'_i\sum_{n=1}^{N} n\sum_{j=1}^{k'_i}\phi_{i,j}\,(S'_{i,n})_j)$
$= \frac{1}{M}(\sum_{i=1}^{k*}\pi'_i\sum_{n=1}^{N} n\sum_{j=1}^{k'_i}\phi_{i,j}\,(S'_i)_j\,({}^i_j q_n)) = \frac{1}{M}(\sum_{i=1}^{k*}\pi'_i\sum_{j=1}^{k'_i}\phi_{i,j}\,(S'_i)_j\, E({}^i_j\psi)$. This gives the stability condition as $\sum_{i=1}^{k*}\pi'_i\sum_{j=1}^{k_i}\varphi_{i,j}\,(T'_i)_j\, E({}^i_j\chi) < \sum_{i=1}^{k*}\pi'_i\sum_{j=1}^{k'_i}\phi_{i,j}\,(S'_i)_j\, E({}^i_j\psi)$ (24)

From the result (24) result for the case of PH/PH/1 bulk queue without environment Ramshankar et al., [8] can be deduced. This result (24) is the stability condition for the random environment PH/PH/1 bulk queue with random sizes of arrivals and of services where maximum arrival size is greater than the maximum service size. When (24) is satisfied, the stationary distribution of the queue length exists Neuts [5]. Let π(0, i, j) : for $1 \le i \le k^*$ $1 \le j \le k_i$; π(0, k, i, j, j') ; for $1 \le k \le M-1$; $1 \le i \le k^*$; $1 \le j \le k_i$; $1 \le j \le k'_i$ and π(n, k, i, j, j'): for $0 \le k \le M-1$; $1 \le i \le k^*$; $1 \le j \le k_i$; $1 \le j \le k'_i$ and $n \ge 1$ be the stationary probability vectors of Markov chain X(t) states.

Let π_0=(π(0,1,1),π(0,1,2),…π(0,1,k_1),π(0,2,1),π(0,2,2),……π(0,2,k_2),……,π(0,k*,1),π(0,k*,2),……π(0,k*,k_{k*}), π(0,1,1,1,1),π(0,1,1,1,2)…π(0,1,1,k_1, k'_1),π(0,1,2,1,1),π(0,1,2,1,2)…π(0,1,2,k_2, k'_2),π(0,1,3,1,1),π(0,1,3,1,2)… π(0,1,3,k_2, k'_2)…π(0,1,k*,1,1),π(0,1,k*,1,2)……π(0,1,k*,k_{k*}, k'_{k*}),π(0,2,1,1,1),……π(0,2,k*,k_{k*}, k'_{k*})……… π(0,M-1,1,1,1),π(0,M-1,1,1,2)……π(0,M-1,k*,k_{k*}, k'_{k*})) be a vector of type 1x$[\sum_{i=1}^{k*} k_i + (M-1)\sum_{i=1}^{k*} k_i\, k'_i]$.

Let π_n=(π(n,0,1,1,1),π(n,0,1,1,2)……π(n,0,1,k_1, k'_1),π(n,0,2,1,1),π(n,0,2,1,2)……π(n,0,2,k_2, k'_2),π(n,0,3,1,1), π(n,0,3,1,2)…π(n,0,3,k_3, k'_3)…π(n,0,k*,1,1),π(n,0,k*,1,2)……π(n,0,k*,k_{k*}, k'_{k*}),π(n,1,1,1,1),π(n,1,1,1,2)…… π(n,1,1,k_1, k'_1),π(n,1,2,1,1),π(n,1,2,1,2)……π(n,1,2,k_2, k'_2),π(n,1,3,1,1),π(n,1,3,1,2)……π(n,1,3,k_3, k'_3)…… π(n,1,k*,1,1),π(n,1,k*,1,2)…π(n,1,k*,k_{k*}, k'_{k*}),π(n,2,1,1,1),…π(n,2,k*,k_{k*}, k'_{k*}),π(n,3,1,1,1)…

π(n,3,k*,k_{k*},k'_{k*})….π(n,M-1,1,1,1),π(n,M-1,1,1,2)………π(n,M-1,k*,k_{k*} ,k'_{k*})) be a vector of type 1x[$M\sum_{i=1}^{k*} k_i\, k'_i$]. The stationary probability vector $\pi = (\pi_0, \pi_1, \pi_3, ...)$ satisfies the equations πQ_A=0 and πe=1. (25)
From (25), it can be seen $\pi_0 B_1 + \pi_1 B_2$=0. (26)
$\pi_0 B_0$+$\pi_1 A_1$+$\pi_2 A_2$ = 0 (27)
$\pi_{n-1} A_0$+$\pi_n A_1$+$\pi_{n+1} A_2$ = 0, for n ≥ 2. (28)
Introducing the rate matrix R as the minimal non-negative solution of the non-linear matrix equation A_0+RA_1+$R^2 A_2$=0, (29)
it can be proved (Neuts [5]) that π_n satisfies the following. $\pi_n = \pi_1 R^{n-1}$ for n ≥ 2. (30)
Using (26), π_0 satisfies $\pi_0 = \pi_1 B_2 (-B_1)^{-1}$ (31)
So using (27) and (31) and (30) the vector π_1 can be calculated up to multiplicative constant since π_1 satisfies the equation $\pi_1\,[B_2(-B_1)^{-1}B_0 + A_1 + RA_2]$ =0. (32)
Using (31) and (30) it can be seen that $\pi_1 [B_2(-B_1)^{-1}$e+(I-R)^{-1}e] = 1. (33)
Replacing the first column of the matrix multiplier of π_1 in equation (32), by the column vector multiplier of π_1 in (33), a matrix which is invertible may be obtained. The first row of the inverse of that same matrix is π_1 and this gives along with (31) and (30) all the stationary probabilities of the system. The matrix R is iterated starting with $R(0) = 0$; and finding $R(n+1)$=$-A_0 A_1^{-1}$–$R^2(n) A_2 A_1^{-1}$, for n ≥ 0. The iteration may be terminated to get a solution of R at a norm level where $\|R(n+1) - R(n)\|$ < ε.

2.3. Performance Measures of the System

(i) The probability of the queue length S = r > 0, P(S=r) can be seen as follows. For 1 ≤ r ≤ M-1, P(S =r) = $\sum_{i=1}^{k*} \sum_{j_1=1}^{k_i} \sum_{j_2=1}^{k'_1} \pi(0, r, i, j_1, j_2)$. For r ≥ M, let n and k be non negative integers such that r = n M + k. Then P(S=r) =$\sum_{i=1}^{k*} \sum_{j_1=1}^{k_i} \sum_{j_2=1}^{k'_i} \pi(n, k, i, j_1, j_2)$, where r = n M + k, n ≥ 1 and k ≥ 0. (34)

(ii) The probability that the queue length is zero is P(S =0) = $\sum_{i=1}^{k*} \sum_{j=1}^{k_i} \pi(0, i, j)$. (35)

(iii) The expected queue level E(S), can be calculated. Using (35) and (34), it may be seen that E(S)=$\sum_0^\infty r\, P(S = r)$=$\sum_{i=1}^{k*} \sum_{j=1}^{k_i} 0\pi(0, i, j)$ +$\sum_{k=1}^{M-1} \sum_{i=1}^{k*} \sum_{j_1=1}^{k_i} \sum_{j_2=1}^{k'_i} k\pi(0, k, i, j_1, j_2)$
+$\sum_{n=1}^{\infty} \sum_{k=0}^{M-1} \sum_{i=1}^{k*} \sum_{j_1=1}^{k_i} \sum_{j_2=1}^{k'_i} \pi(n, k, i, j_1, j_2)(nM + k)$
=$\sum_{k=1}^{M-1} \sum_{i=1}^{k*} \sum_{j_1=1}^{k_i} \sum_{j_2=1}^{k'_i} k\pi(0, k, i, j_1, j_2)$ + $\sum_{n=1}^{\infty} \pi_n$.(Mn,…,Mn,Mn+1,…,Mn+1,Mn+2,…,Mn+2,…, Mn+M-1,…,Mn+M-1) = $\sum_{k=1}^{M-1} k \sum_{i=1}^{k*} \sum_{j_1=1}^{k_i} \sum_{j_2=1}^{k'_i} \pi(0, k, i, j_1, j_2)$ + M$\sum_{n=1}^{\infty} n\pi_n$ e+$\pi_1 (I - R)^{-1}\xi$.
Here ξ=$(0, ... 0, 1, ..., 1, 2, ..., 2, ..., M-1, ..., M-1)'$ is of type [($\sum_{i=1}^{k*} k_i k'_i$)M]x1 column vector in which consecutively ($\sum_{i=1}^{k*} k_i k'_i$) times 0,1,2,3.., M-1 appear. Let it be called ξ' when 0 appears ($\sum_{i=1}^{k*} k_i$)times and others in that order appear ($\sum_{i=1}^{k*} k_i k'_i$) times.
Then E(S)= π_0ξ'+ $\pi_1 (I - R)^{-1}\xi + M\pi_1 (I - R)^{-2} e$ (36)
(iv)Variance of S can be derived. Let η be column vector η=[0,..,0, 1^2, ... 1^2 2^2,.., 2^2, ... $(M-1)^2, ..., (M-$ *1)2'* of type [(*i=1k∗kik'i*)M]x1in which consecutively (*i=1k∗kik'i)* times squares of 0,1,2,3.., M-1 appear. Let it be called η' when 0 appears ($\sum_{i=1}^{k*} k_i$) times and others in the same manner as in η appear ($\sum_{i=1}^{k*} k_i k'_i$) times. Then it can be seen that the second moment,
E(S^2)=$\sum_0^\infty r^2\, P(S = r)$=$\sum_{i=1}^{k*} \sum_{j=1}^{k_i} 0\pi(0, i, j)$ +$\sum_{k=1}^{M-1} \sum_{i=1}^{k*} \sum_{j_1=1}^{k_i} \sum_{j_2=1}^{k'_i} \pi(0, k, i, j_1, j_2) k^2$
+$\sum_{n=1}^{\infty} \sum_{k=0}^{M-1} \sum_{i=1}^{k*} \sum_{j_1=1}^{k_i} \sum_{j_2=1}^{k'_i} \pi(n, k, i, j_1, j_2)(nM + k)^2$ =π_0η' + $M^2 [\sum_{n=1}^{\infty} n(n-1)\pi_n\, e + \sum_{n=1}^{\infty} n\pi_n\, e]$ + $\sum_{n=1}^{\infty} \pi_n \eta$ + 2M $\sum_{n=1}^{\infty} n\, \pi_n\, \xi$.
So, E(S^2)= π_0η'+$M^2 [\pi_1 (I - R)^{-3} 2R\, e + \pi_1 (I - R)^{-2} e] + \pi_1 (I - R)^{-1}\eta + 2M\pi_1 (I - R)^{-2}\xi$ (37)
VAR(S)=E(S^2) $- [E(S)]^2$may be written from (36) and(37).

III. MODEL (B) MAXIMUM ARRIVAL SIZE M < MAXIMUM SERVICE SIZE N

The dual case of Model (A), namely the case, M < N is treated here. (When M =N both models are applicable and one can use any one of them.) The assumption (vii) of Model (A) is changed and all its other assumptions are retained.

3.1.Assumption

(vii) The maximum arrival size M= $\max_{1 \le i \le k*} \max_{1 \le j \le k_i} {}_j^i M$ is less than the maximum service size N=$\max_{1 \le i \le k*} \max_{1 \le j \le k'_i} {}_j^i N$.

3.2.Analysis

Since this model is dual, the analysis is similar to that of Model (A). The differences are noted below. The state space of the chain is as follows presented in a similar way.

The state of the system of the continuous time Markov chain X (t) under consideration is presented as follows. X(t) = {(0, i, j) : for 1 ≤ i ≤ k* 1 ≤ j ≤ k_i)} U {(0, k, i, j, j') ; for 1 ≤ k ≤ N-1; 1 ≤ i ≤ k*; 1 ≤ j ≤ k_i; 1 ≤ j ≤ k'_i} U {(n, k, i, j, j'): for 0 ≤ k ≤ N-1; 1 ≤ i ≤ k*; 1 ≤ j ≤ k_i; 1 ≤ j ≤ k'_i and n ≥ 1}. (38)

The chain is in the state (0, i, j) when the number of customers in the queue is 0, the environment state is i for 1 ≤ i ≤ k*and the arrival phase is j for 1 ≤ j ≤ k_i. The chain is in the state (0, k, i, j, j') when the number of customers is k for 1 ≤ k ≤ N-1, the environment state is i for 1 ≤ i ≤ k*, the arrival phase is j for 1 ≤ j ≤ k_i and the service phase is j' for 1 ≤ j' ≤ k'_i. The chain is in the state (n, k, i, j, j') when the number of customers in the queue is n N + k, for 0 ≤ k ≤ N-1 and 1 ≤ n < ∞, the environment state is i for 1 ≤ i ≤ k*, the arrival phase is j for 1 ≤ j ≤ k_i and the service phase is j' for 1 ≤ j' ≤ k'_i. When the number of customers waiting in the system is r, then r is identified with (n, k) where r on division by N gives n as the quotient and k as the remainder. The infinitesimal generator Q_B of the model has the same structure given in (4) but the inner matrices are of different orders.

$$Q_B=\begin{bmatrix} B'_1 & B'_0 & 0 & 0 & . & . & . & \cdots \\ B'_2 & A'_1 & A'_0 & 0 & . & . & . & \cdots \\ 0 & A'_2 & A'_1 & A'_0 & 0 & . & . & \cdots \\ 0 & 0 & A'_2 & A'_1 & A'_0 & 0 & . & \cdots \\ 0 & 0 & 0 & A'_2 & A'_1 & A'_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \quad (39)$$

In (39) the states of the matrices are listed lexicographically as $\underline{0}, \underline{1}, \underline{2}, \underline{3}, \ldots$. For partition purpose the zero states in the first two sets of (38) are combined. The vector $\underline{0}$ is of type 1 x $[\sum_{i=1}^{k*} k_i + (N-1)\sum_{i=1}^{k*} k_i\, k'_i]$ and is $\underline{0}$=((0,1,1),(0,1,2),…(0,1, k_1),(0,2,1),(0,2,2),…(0,2, k_2),……(0,k*,1),(0,k*,2),…(0,k*,k_{k*}),(0,1,1,1,1),(0,1,1,1,2) ….(0,1,1,1, k'_1),(0,1,1,2,1),(0,1,1,2,2)….(0,1,1,2,k'_1),(0,1,1,3,1)....(0,1,1,3,k'_1)…..(0,1,1,k_1,1)… (0,1,1,k_1, k'_1),(0,1,2,1,1),(0,1,2,1,2)….(0,1,2,1, k'_2),(0,1,2,2,1),(0,1,2,2,2)….(0,1,2,2,k'_2),(0,1,2,3,1)....(0,1,2,3, k'_2)….(0,1,2,k_2,1)…(0,1,2,k_2, k'_2),(0,1,3,1,1)…(0,1,3,k_3, k'_3)…(0,1,k*,1,1),….(0,1,k*,k_{k*}, k'_{k*}),(0,2,1,1,1), (0,2,1,1,2)…(0,2,k*,k_{k*},k'_{k*}),(0,3,1,1,1)…(0,3,k*,$k_{k*}k_{k*}$),(0,4,1,1,1)…(0,4,k*,$k_{k*}k'_{k*}$)…(0,N-1,1,1,1)… (0,N-1,k*,k_{k*}, k'_{k*})) and the vector $\underline{n}$ is of type 1x$[N\sum_{i=1}^{k*} k_i\, k'_i]$ and is given in a similar manner as follows $\underline{n}$=(n,0,1,1,1),(n,0,1,1,2)….(n,0,1,1, k'_1),(n,0,1,2,1),…(n,0,1,2,k'_1),…(n,0,k*,1,1),…(n,0,k*,k_{k*}, k'_{k*}),(n,1,1,1,1) …. (n,1,k*,k_{k*},k'_{k*}),(n,2,1,1,1)....(n,2,k*,k_{k*},k'_{k*})……..(n,N-1,1,1,1),(n,N-1,1,1,2)……(n,N-1,k*,k_{k*}, k'_{k*})).

The matrices$B'_1 and\ A'_1$ have negative diagonal elements, they are of orders $[\sum_{i=1}^{k*} k_i + (N-1)\sum_{i=1}^{k*} k_i\, k'_i]$ and $[N\sum_{i=1}^{k*} k_i\, k'_i]$ respectively and their off diagonal elements are non-negative. The matrices $A'_0\ and A'_2$ have nonnegative elements and are of order $[N\sum_{i=1}^{k*} k_i\, k'_i]$. The matrices $B'_0\ and\ B'_2$ have non-negative elements and are of types $[\sum_{i=1}^{k*} k_i + (N-1)\sum_{i=1}^{k*} k_i\, k'_i]$ x $[N\sum_{i=1}^{k*} k_i\, k'_i]$ and $[N\sum_{i=1}^{k*} k_i\, k'_i]$ x $[\sum_{i=1}^{k*} k_i + (N-1)\sum_{i=1}^{k*} k_i\, k'_i]$ and they are given below. Using Model (A) for definitions of $\Lambda_j\ and \Lambda'_j, and\ \ U_j, V_j$, and U and letting Ω and Ω' as in Model (A), the partitioning matrices are defined as follows. The matrix $B'_0$is same as that of A'_0 with first zero block row is of order$[\sum_{i=1}^{k*} k_i]$ x$[N\sum_{i=1}^{k*} k_i\, k'_i]$. The matrix $B'_2$is same as that of A'_2 except the first column block is of type $[N\sum_{i=1}^{k*} k_i\, k'_i]$ x $[\sum_{i=1}^{k*} k_i]$ and is $(U'_N, 0, \ldots .0)'$ where

$$U'_N=\begin{bmatrix} I_{k_1}\otimes S'_{1,N} & 0 & 0 & \cdots & 0 \\ 0 & I_{k_2}\otimes S'_{2,N} & 0 & \cdots & 0 \\ 0 & 0 & I_{k_3}\otimes S'_{3,N} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_{k_{k*}}\otimes S'_{k*,N} \end{bmatrix} \quad (40)$$

$$A'_0=\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \Lambda_M & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Lambda_2 & \Lambda_3 & \cdots & \Lambda_M & 0 & 0 & \cdots & 0 \\ \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-1} & \Lambda_M & 0 & \cdots & 0 \end{bmatrix} \quad (41)$$

$$A'_2 = \begin{bmatrix} U_N & U_{N-1} & U_{N-2} & \cdots & U_3 & U_2 & U_1 \\ 0 & U_N & U_{N-1} & \cdots & U_4 & U_3 & U_2 \\ 0 & 0 & U_N & \cdots & U_5 & U_4 & U_3 \\ 0 & 0 & 0 & \ddots & U_6 & U_5 & U_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & U_N & U_{N-1} & U_{N-2} \\ 0 & 0 & 0 & \cdots & 0 & U_N & U_{N-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & U_N \end{bmatrix} \tag{42}$$

$$A'_1 = \begin{bmatrix} \Omega & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_M & 0 & 0 & \cdots & 0 & 0 \\ U_1 & \Omega & \Lambda_1 & \cdots & \Lambda_{M-1} & \Lambda_M & 0 & \cdots & 0 & 0 \\ U_2 & U_1 & \Omega & \cdots & \Lambda_{M-2} & \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ U_{N-M-1} & U_{N-M-2} & U_{N-M-3} & \cdots & \Omega & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-1} & \Lambda_M \\ U_{N-M} & U_{N-M-1} & U_{N-M-2} & \cdots & U_1 & \Omega & \Lambda_1 & \cdots & \Lambda_{M-2} & \Lambda_{M-1} \\ U_{N-M+1} & U_{N-M} & U_{N-M-1} & \cdots & U_2 & U_1 & \Omega & \cdots & \Lambda_{M-3} & \Lambda_{M-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ U_{N-2} & U_{N-3} & U_{N-4} & \cdots & U_{N-M-2} & U_{N-M-3} & U_{N-M-2} & \cdots & \Omega & \Lambda_1 \\ U_{N-1} & U_{N-2} & U_{N-3} & \cdots & U_{N-M-1} & U_{N-M-2} & U_{N-M-1} & \cdots & U_1 & \Omega \end{bmatrix} \tag{43}$$

$$B'_1 = \begin{bmatrix} \Omega' & \Lambda'_1 & \Lambda'_2 & \cdots & \Lambda'_M & 0 & 0 & \cdots & 0 & 0 \\ U & \Omega & \Lambda_1 & \cdots & \Lambda_{M-1} & \Lambda_M & 0 & \cdots & 0 & 0 \\ V_1 & U_1 & \Omega & \cdots & \Lambda_{M-2} & \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ V_{N-M-2} & U_{N-M-2} & U_{N-M-3} & \cdots & \Omega & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-1} & \Lambda_M \\ V_{N-M-1} & U_{N-M-1} & U_{N-M-2} & \cdots & U_1 & \Omega & \Lambda_1 & \cdots & \Lambda_{M-2} & \Lambda_{M-1} \\ V_{N-M} & U_{N-M} & U_{N-M-1} & \cdots & U_2 & U_1 & \Omega & \cdots & \Lambda_{M-3} & \Lambda_{M-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ V_{N-3} & U_{N-3} & U_{N-4} & \cdots & U_{N-M-2} & U_{N-M-3} & U_{N-M-2} & \cdots & \Omega & \Lambda_1 \\ V_{N-2} & U_{N-2} & U_{N-3} & \cdots & U_{N-M-1} & U_{N-M-2} & U_{N-M-1} & \cdots & U_1 & \Omega \end{bmatrix} \tag{44}$$

$$\mathcal{Q}''_B = \begin{bmatrix} \Omega + U_N & \Lambda_1 + U_{N-1} & \cdots & \Lambda_{M-1} + U_{N-M+1} & \Lambda_M + U_{N-M} & U_{N-M-1} & \cdots & U_2 & U_1 \\ U_1 & \Omega + U_N & \cdots & \Lambda_{M-2} + U_{N-M+2} & \Lambda_{M-1} + U_{N-M+1} & \Lambda_M + U_{N-M} & \cdots & U_3 & U_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ U_{N-M-2} & U_{N-M-3} & \cdots & \Omega + U_N & \Lambda_1 + U_{N-1} & \Lambda_2 + U_{N-2} & \cdots & \Lambda_M + U_{N-M} & U_{N-M-1} \\ U_{N-M-1} & U_{N-M-2} & \cdots & U_1 & \Omega + U_N & \Lambda_1 + U_{N-1} & \cdots & \Lambda_{M-1} + U_{N-M+1} & \Lambda_M + U_{N-M} \\ \Lambda_M + U_{N-M} & U_{N-M-1} & \cdots & U_2 & U_1 & \Omega + U_N & \cdots & \Lambda_{M-2} + U_{N-M+2} & \Lambda_{M-1} + U_{N-M+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Lambda_2 + U_{N-2} & \Lambda_3 + U_{N-3} & \cdots & U_{N-M-1} & U_{N-M-2} & U_{N-M-3} & \cdots & \Omega + U_N & \Lambda_1 + U_{N-1} \\ \Lambda_1 + U_{N-1} & \Lambda_2 + U_{N-2} & \cdots & \Lambda_M + U_{N-M} & U_{N-M-1} & U_{N-M-2} & \cdots & U_1 & \Omega + U_N \end{bmatrix} \tag{45}$$

The basic generator which is concerned with only the arrival and service is $\mathcal{Q}''_B = A'_0 + A'_1 + A'_2$. This is also block circulant. Using similar arguments given for Model (A) it can be seen that its probability vector is w' = $\left(\frac{w}{N}, \frac{w}{N}, \frac{w}{N}, \ldots\ldots, \frac{w}{N}\right)$ where w is given by w = ($\pi'_1(\varphi_1 \otimes \phi_1)$, $\pi'_2(\varphi_2 \otimes \phi_2)$,……, $\pi'_{k*}(\varphi_{k*} \otimes \phi_{k*})$) and the stability condition remains the same. Following the arguments given for Model (A), one can find the stationary probability vector for Model (B) also in matrix geometric form. All performance measures including expectation of customers waiting for service and its variance for Model (B) have the form as in Model (A) except M is replaced by N.

IV. NUMERICAL ILLUSTRATIONS

Numerical cases are presented here to illustrate the application of the study. Three examples are studied namely (i) M=N=3 (ii) M=3, N=2 and (iii) M=2, N=3. The environment has two states governed by the Markov chain with infinitesimal generator $\begin{bmatrix} -3 & 3 \\ 5 & -5 \end{bmatrix}$. In state 1, the arrival time distribution is PH 1 (exponential) with parameter 5 and the service time distribution is PH 2with representation $\begin{bmatrix} -3 & 1 \\ 2 & -3 \end{bmatrix}$ and starting probability vector (.6, .4). In environment 2 the arrival time has PH 2 distribution with representation $\begin{bmatrix} -2 & 1 \\ 2 & -4 \end{bmatrix}$ and starting probability vector (.4, .6) and the service time distribution is PH 1 (exponential) with parameter 8. For the case (i) M=N=3, in the environment 1, the probabilities of bulk arrivals are taken as ${}^{1}_{1}p_1$=.6, ${}^{1}_{1}p_2$=.2 and ${}^{1}_{1}p_3$=.2 and the probabilities of bulk services in the two phases are considered as ${}^{1}_{1}q$=.5, ${}^{1}_{1}q_2$=.3, ${}^{1}_{1}q_3$=.2, ${}^{1}_{2}q_1$=.4, ${}^{1}_{2}q_2$=.6 and ${}^{1}_{2}q_3$=0. In the environment 2, the bulk arrival probabilities are assumed as ${}^{2}_{1}p_1$=.4, ${}^{2}_{1}p_2$=.6, ${}^{2}_{1}p_3$=0, ${}^{2}_{2}p_1$=.5, ${}^{2}_{2}p_2$=.5 and ${}^{2}_{2}p_3$=0 and bulk service probabilities are taken as ${}^{2}_{2}q_1$=.5, ${}^{2}_{2}q_2$=.5 and ${}^{2}_{2}q_3$=0. For the case (ii) M=3, N =2 only two probabilities of case (i) alone are changed namely, ${}^{1}_{1}q_2$=.5 and ${}^{1}_{1}q_3$=0, keeping the same values of case (i) for all others. For the case (iii) M=2, N=3, the probabilities of case (i) are assumed changing only two values namely, ${}_1^1p_2$=.4 and ${}_1^1p_3 = 0$. The rate matrix R is taken for calculation after 30 iterations for the three models. The results obtained are presented in the table 1 below. The probabilities and expected values show significant variation for higher and lower values of M and N. The figures (1) and (2) show the variations of probabilities.

Table 1. Results Obtained for the Three Cases.

	M=N=3	M=3;N=2	M=2;N=3
P(S=0)	0.060354629	0.050226694	0.115130656
P(S=1)	0.027940388	0.02353663	0.053013985
P(S=2)	0.026006538	0.022495782	0.04491614
P(S=3)	0.025331264	0.022031479	0.042021471
P(S=4)	0.024460863	0.021384356	0.039162694
P(S=5)	0.0236716	0.020802411	0.036546073
π0e	0.116536161	0.097720416	0.227289746
π1e	0.081693776	0.070317631	0.144526435
π2e	0.073463727	0.064218246	0.117730239
π3e	0.06669037	0.059222531	0.095641597
π4e	0.060580626	0.054651446	0.077721784
π(n>4)e	0.601035341	0.653869731	0.337090199
Norm	0.000630675	0.000681993	0.00032906
Arri rate	1.694444444	1.694444444	1.518518519
Serv rate	2.462962963	2.433333333	2.462962963
E(S)	29.86540309	36.01090259	13.28458975
Std dev(S)	31.21206209	37.35092795	14.41652322

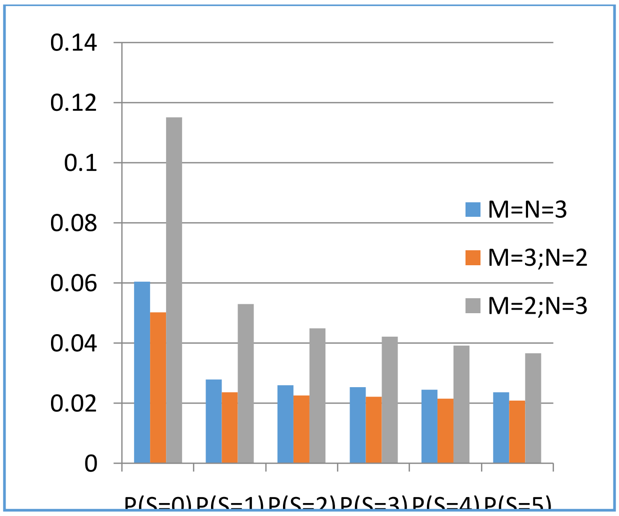

Figure 1. Probabilities of Queue Lengths

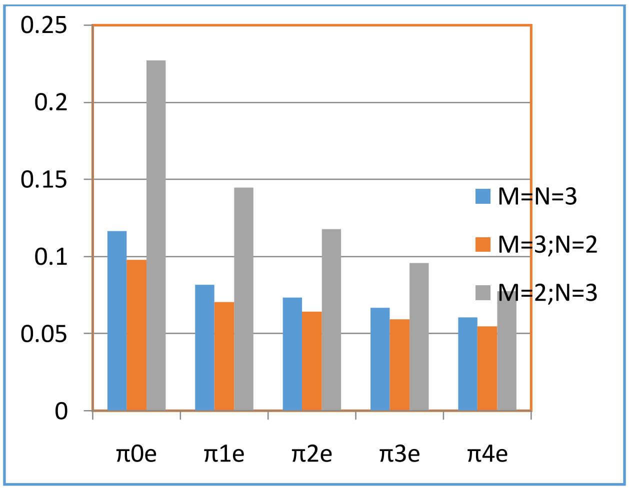

Figure2.The Probabilities of Customer Blocks

V. CONCLUSION

Two PH/PH/1 bulk arrival and bulk service queues with random environment have been studied by identifying the maximum of the arrival and service sizes and grouping the customers as members of blocks of such maximum sizes. Matrix geometric results have been obtained by partitioning the infinitesimal generator by grouping of customers, environment state and PH phases together. The basic system generators of the queues are block circulant matrices which are explicitly presenting the stability condition in standard forms. Numerical results for bulk queue models are presented and discussed. Effects of variation of rates on expected queue length and on probabilities of queue lengths are exhibited. The decrease in arrival rates (so also increase in service rates) makes the convergence of R matrix faster which can be seen in the decrease of norm values. The standard deviations also decrease. The PH/PH/1 queue with bulk arrival and bulk service with random environment has number of applications. The PH distributions include Exponential, Erlang, Hyper Exponential, and Coxian distributions as special cases and the PH distribution is also a best approximation for a general distribution. Further the PH/PH/1 queue is a most general form almost equivalent to G/G/1 queue. The bulk arrival models because they have non zero elements or blocks above the super diagonals in infinitesimal generators, they require for studies the decomposition methods with which queue length probabilities of the system are written in a recursive manner. Their applications are much limited compared to matrix geometric results. From the results obtained here, provided the maximum arrival and service sizes are not infinity, the most general model of the PH/PH/1 bulk arrivals and bulk services queue with random environment admits matrix geometric solution. Further studies with block circulant basic generator system may produce interesting and useful results in inventory theory and finite storage models like dam theory. It is also noticed here that once the maximum arrival or service size increases, the order of the rate matrix increases proportionally. However the matrix geometric structure is retained and rates of convergence is not much affected. Models with multiple servers with PH distributions may be focused for further study which may produce more general results.

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